Securing Elections using Homomorphic Cryptography

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Voting

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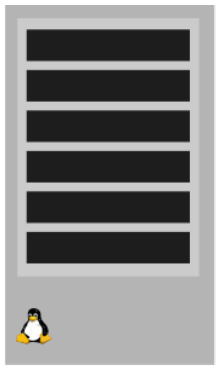
- why is that?

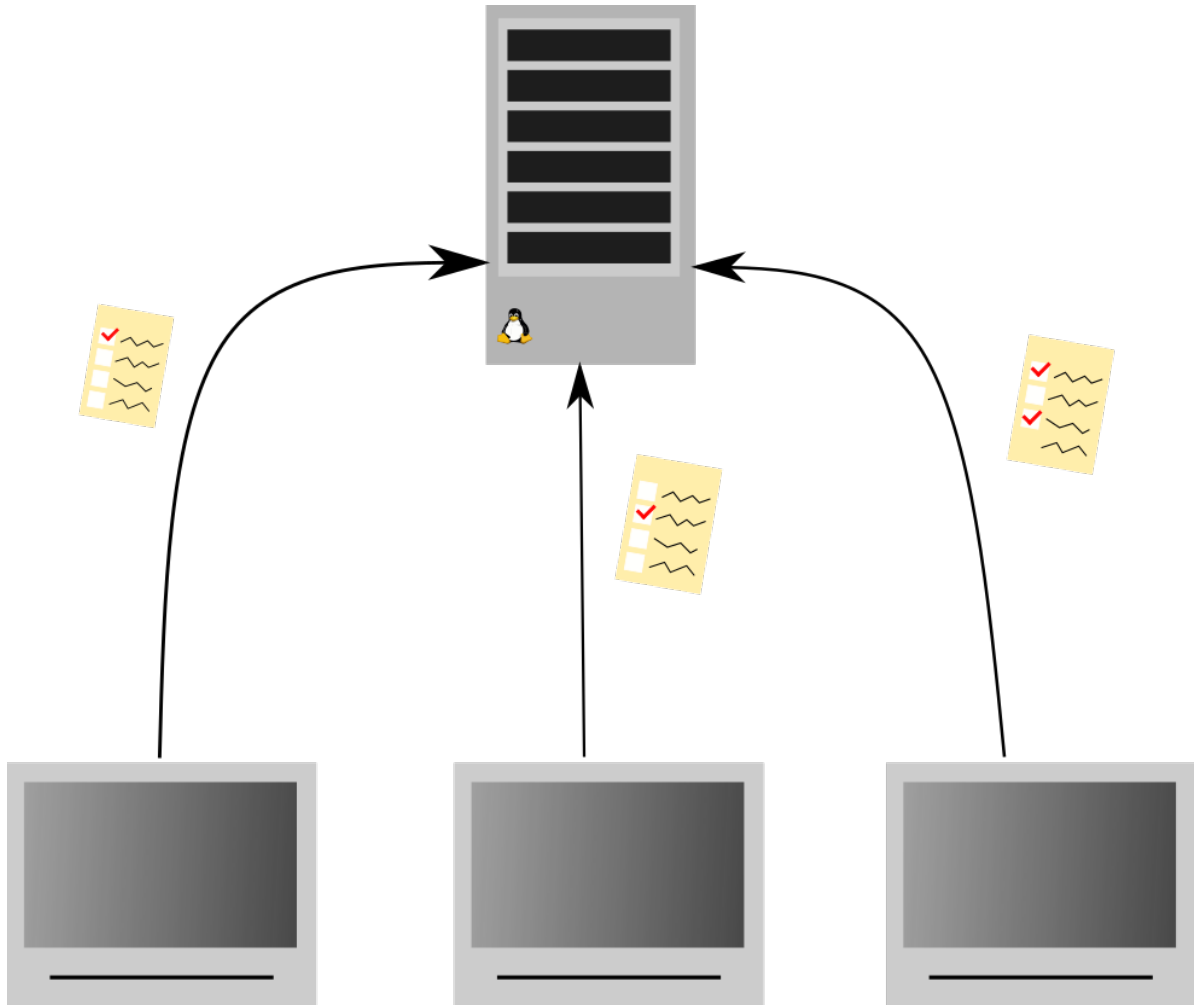
The Traditional Approach in Electronic Terms I

1. voters register with a server
2. once authenticated, voters send their ballots to the server
3. the server counts the votes
4. the aggregate result is published

The Traditional Approach in Electronic Terms II



The Traditional Approach in Electronic Terms III



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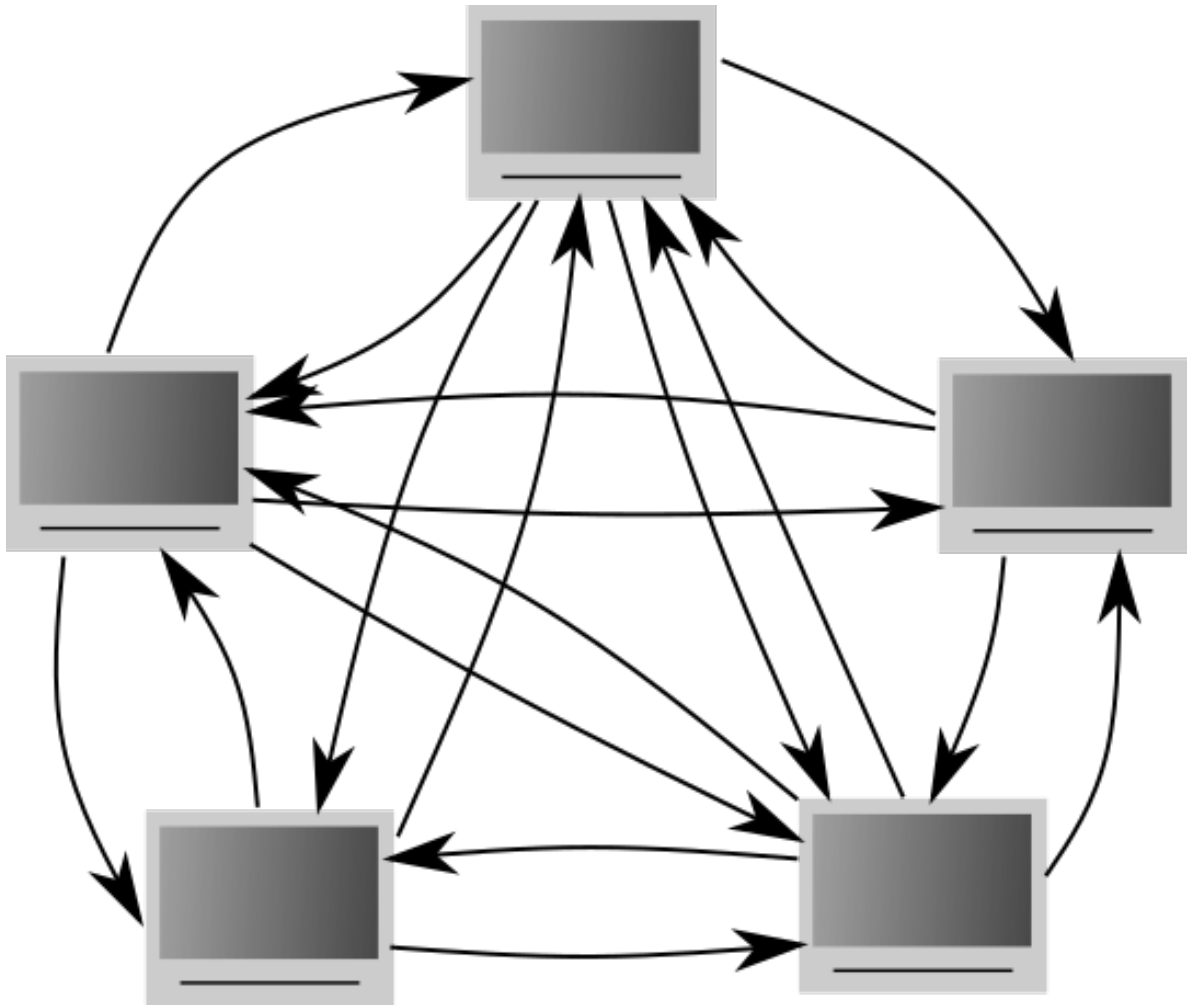
Disclaimer: I did not come up with any of this.

A New Approach I

- instead of a client-server approach, let's have a peer-to-peer (p2p) network where voters connect to each other directly

A New Approach II



- Authentication is essentially the same as before: voters use standard means to identify themselves (think passwords, kerberos, LDAP, SSH/GPG finderprints, multipass, etc.)

The big question

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- **Answer**: we "share" pieces of our secret ballot with everyone!

Introducing secret sharing

- ***Secret sharing*** *(also called secret splitting) refers to methods for distributing a secret amongst a group of participants, each of whom is allocated a share of the secret. The secret can be reconstructed only when a sufficient number, of possibly different types, of shares are combined together; individual shares are of no use on their own.* [1]

[1] `https://en.wikipedia.org/wiki/Secret_sharing`

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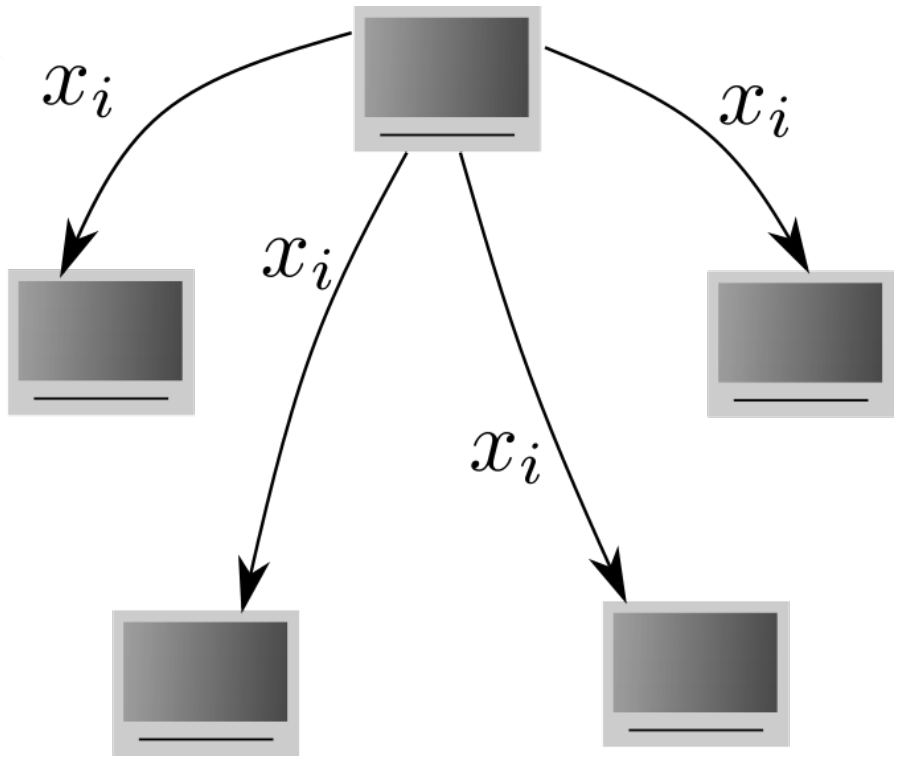

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$$P_i(x) = c_i + a_{(i,1)}x + a_{(i,2)}x^2 + \cdots + a_{(i,k-1)}x^{k-1}$$

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- Every $a_{(i,j)}$ is a random natural number.
- c_i is the numerically-encoded contents of voter i's ballot. Every c_i must be encoded in such a way that summing every $\{c_j\}_{j\in[1,k]}$ results in the desired election result. For example, in a simple "yes/no" election, 1 could represent "yes" and -1 could represent "no".

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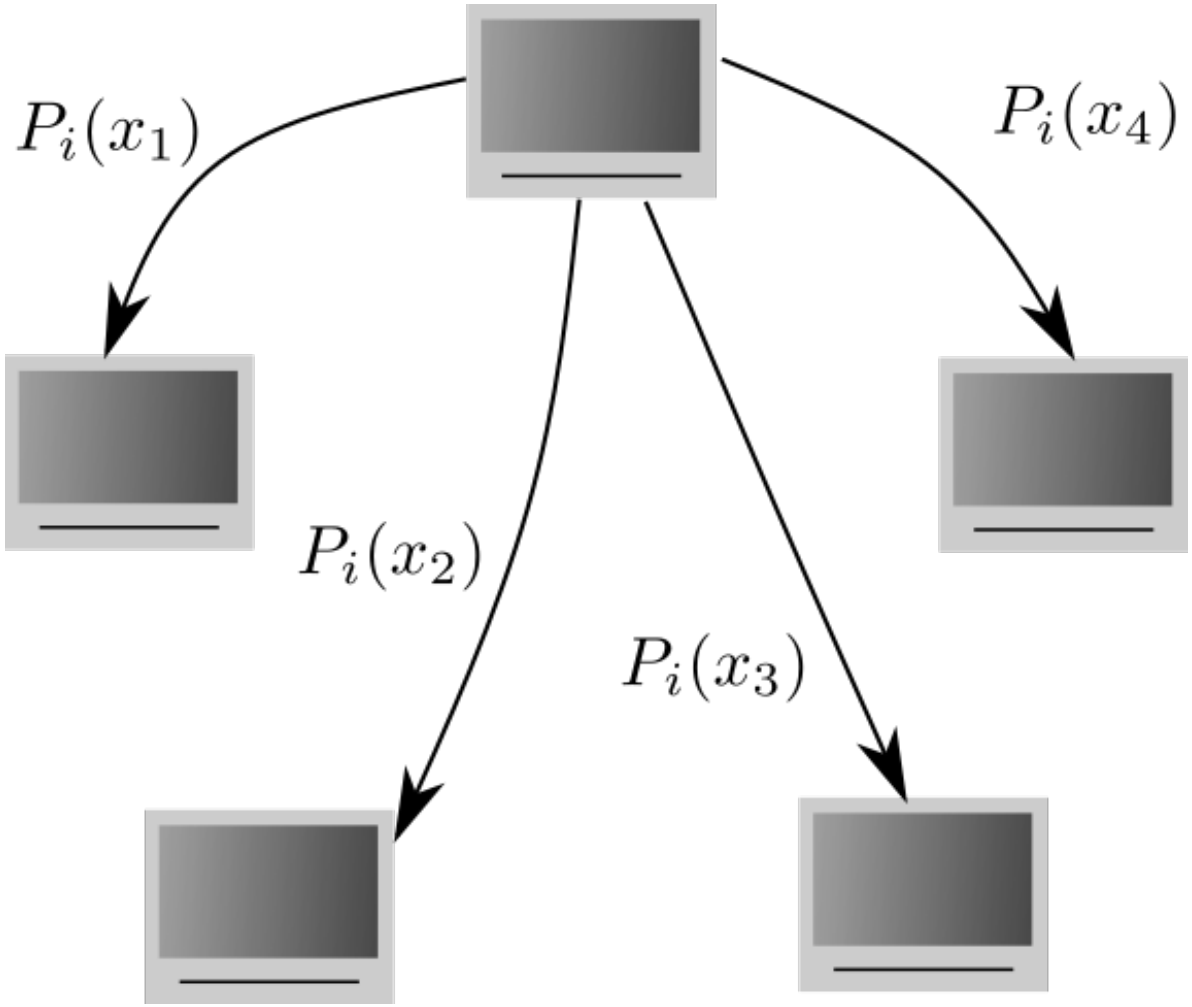
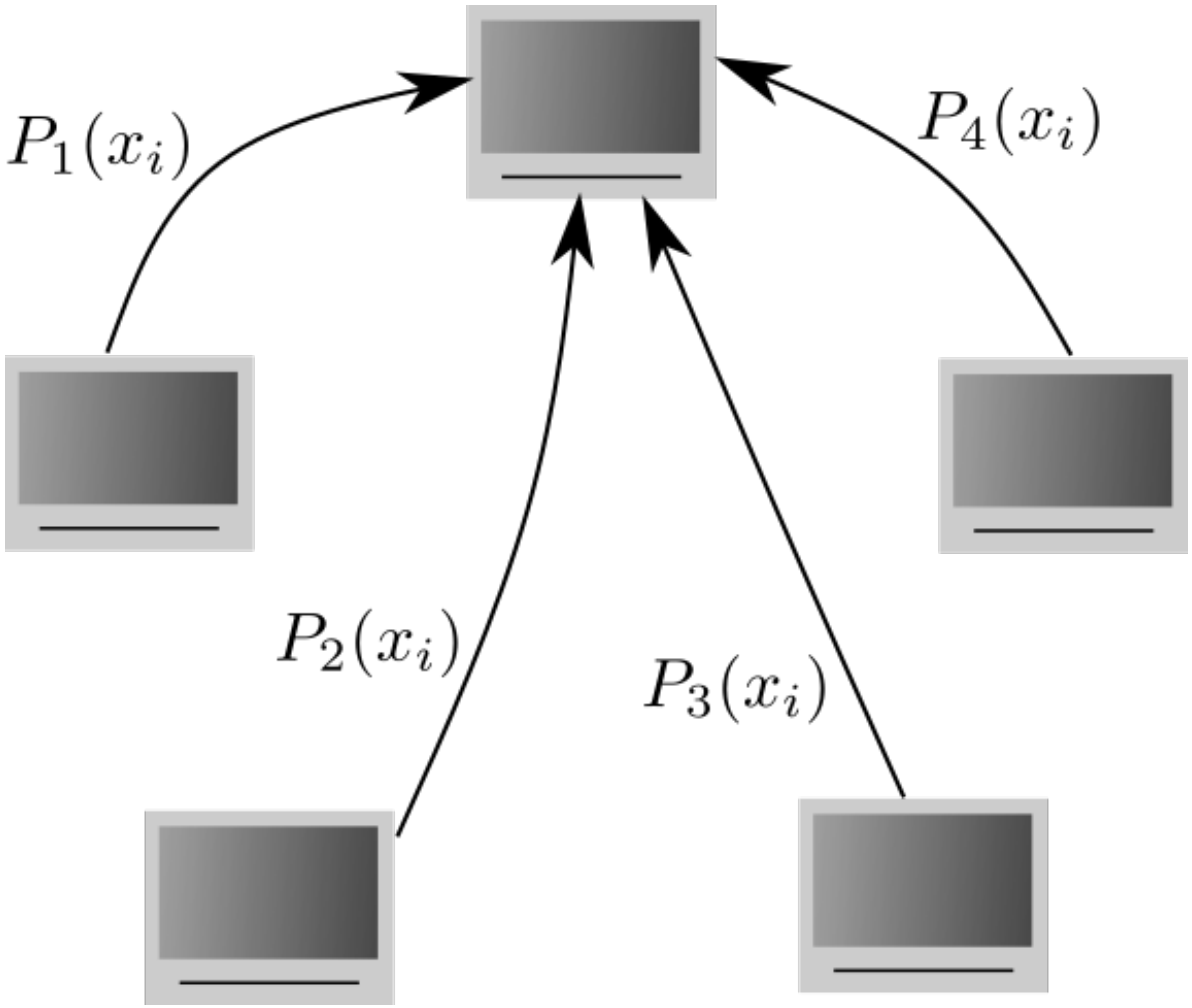
- Thus, knowing everything about our $k-1$ degree polynomial P_i is equivalent to knowing k points on it.
- So, we're going to distribute our secret by giving everyone just one point.

- Every voter sends every other voter the result of evulating their polynomial at the given voter's public input:



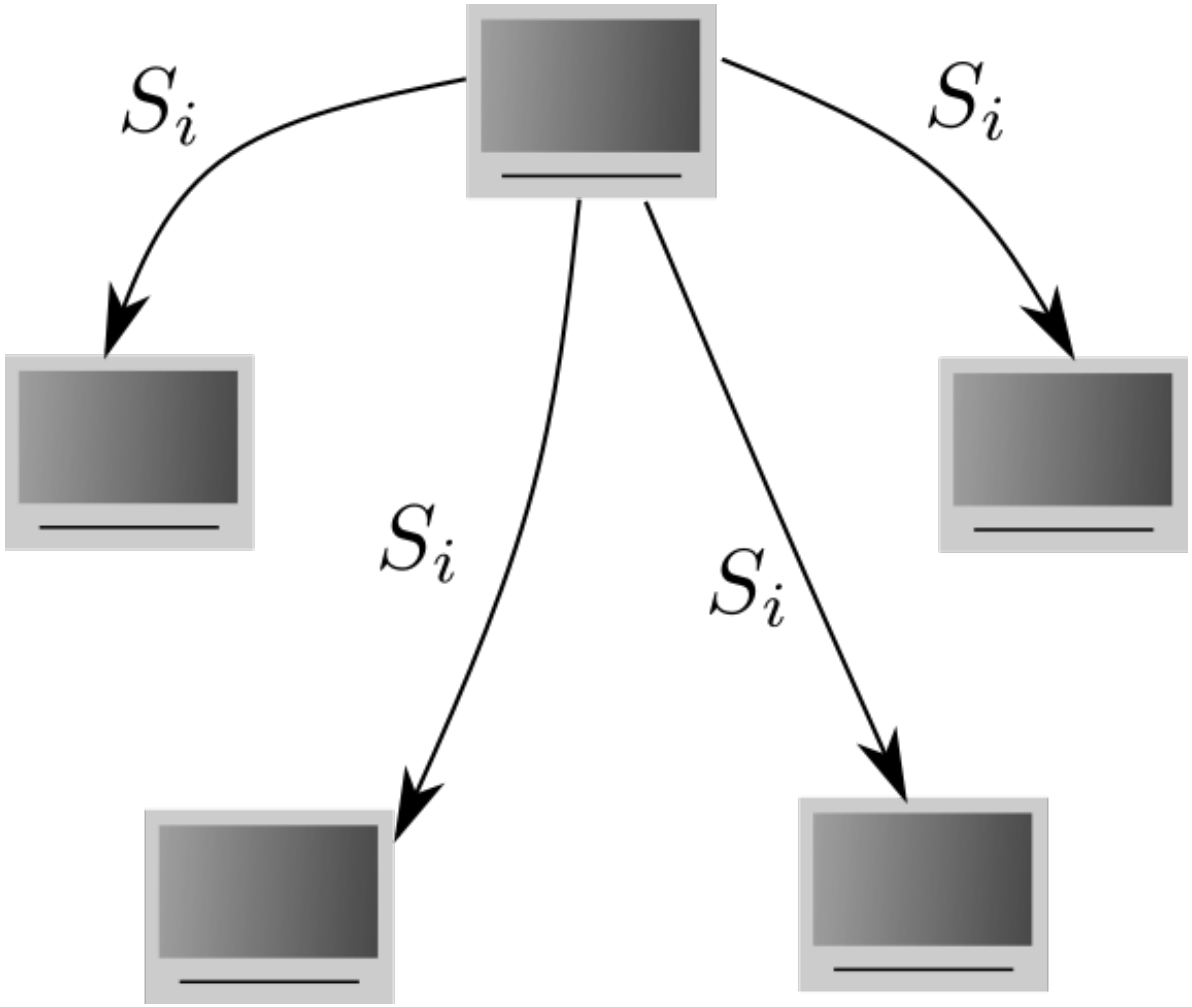

$P_1(x_i)$
$P_4(x_i)$
$P_2(x_i)$
$P_3(x_i)$

- Voter i now has:
  - $P_1(x_i)$
  - $P_2(x_i)$
  - $\vdots$
  - $P_k(x_i)$

Let's sum those values!

$$
\begin{aligned}
S_i =& P_1(x_i) + P_2(x_i) + \cdots + P_k(x_i) \\
=& \left(c_1 + a_{(1,1)}x_i + a_{(1,2)}x_i^2 + \cdots + a_{(1,k-1)}x_i^{k-1}\right) + \\
& \left(c_2 + a_{(2,1)}x_i + a_{(2,2)}x_i^2 + \cdots + a_{(2,k-1)}x_i^{k-1}\right) + \\
& \cdots \\
& \left(c_k + a_{(k,1)}x_i + a_{(k,2)}x_i^2 + \cdots + a_{(k,k-1)}x_i^{k-1}\right) \\
=& \sum_{j=1}^{k} c_j + x_i \sum_{j=1}^{k} a_{(j,1)} + x_i^2 \sum_{j=1}^{k} a_{(j,2)} + \cdots + x_i^{k-1} \sum_{j=1}^{k} a_{(j,2)}
\end{aligned}
$$

Let's have every voter share their sum with everyone else.



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$$S_1 = \sum_{j=1}^{k} c_j + x_1 \sum_{j=1}^{k} a_{(j,1)} + x_1^2 \sum_{j=1}^{k} a_{(j,2)} + \cdots + x_1^{k-1} \sum_{j=1}^{k} a_{(j,k-1)}$$

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$$\vdots$$

$$S_k = \sum_{j=1}^{k} c_j + x_k \sum_{j=1}^{k} a_{(j,1)} + x_k^2 \sum_{j=1}^{k} a_{(j,2)} + \cdots + x_k^{k-1} \sum_{j=1}^{k} a_{(j,k-1)}$$

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- These are k points all on the same $k-1$ degree polynomial:

$$(x_1, S_1), (x_2, S_2), \ldots, (x_k, S_k)$$

- Since we know k points of the $k-1$ degree polynomial, we can find the coefficients and constant term of it with a little linear algebra:

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$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} \\ \vdots & & & \ddots & \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-1} \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{k} c_j \\ \sum_{j=1}^{k} a_{(j,1)} \\ \sum_{j=1}^{k} a_{(j,2)} \\ \vdots \\ \sum_{j=1}^{k} a_{(j,k-1)} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_k \end{bmatrix}$$

$$\text{RREF}\left(\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} & S_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} & S_2 \\ \vdots & & & \ddots & & \\ 1 & x_k & x_k^2 & \cdots & x_k^{k-1} & S_k \end{bmatrix}\right)$$

Notice that the constant term of the final polynomial is $\sum_{j=1}^{k} c_j$, which is precisely the result of the election!

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5. All k sums the voters compute are used to find the constant term of a polynomial whose constant term is precisely the sum of all ballot.

Demo

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For example, in a "yes/no" election, someone could put 2 for their ballot and have the result of 2 votes.

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The following material was very rushed.

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- Now, let $C(x)$ be a *constraint checking polynomial* that is zero if x is a valid constant and nonzero otherwise. For example, if we assume a valid constant is one that is either a zero or one, we can construct $C(x)$ very simply:

$$C(x) = (x - 0)(x - 1) = x^2 - x$$

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- Now, we can restate the problem as: we need to prove that $C(P(x)) = 0$ when $x = 0$.

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- Before anything, everyone "commits" [2] to their polynomial by creating a merkle tree of the outputs of $P(x)$ and $D(x)$ values and sending the root of the tree to everyone.

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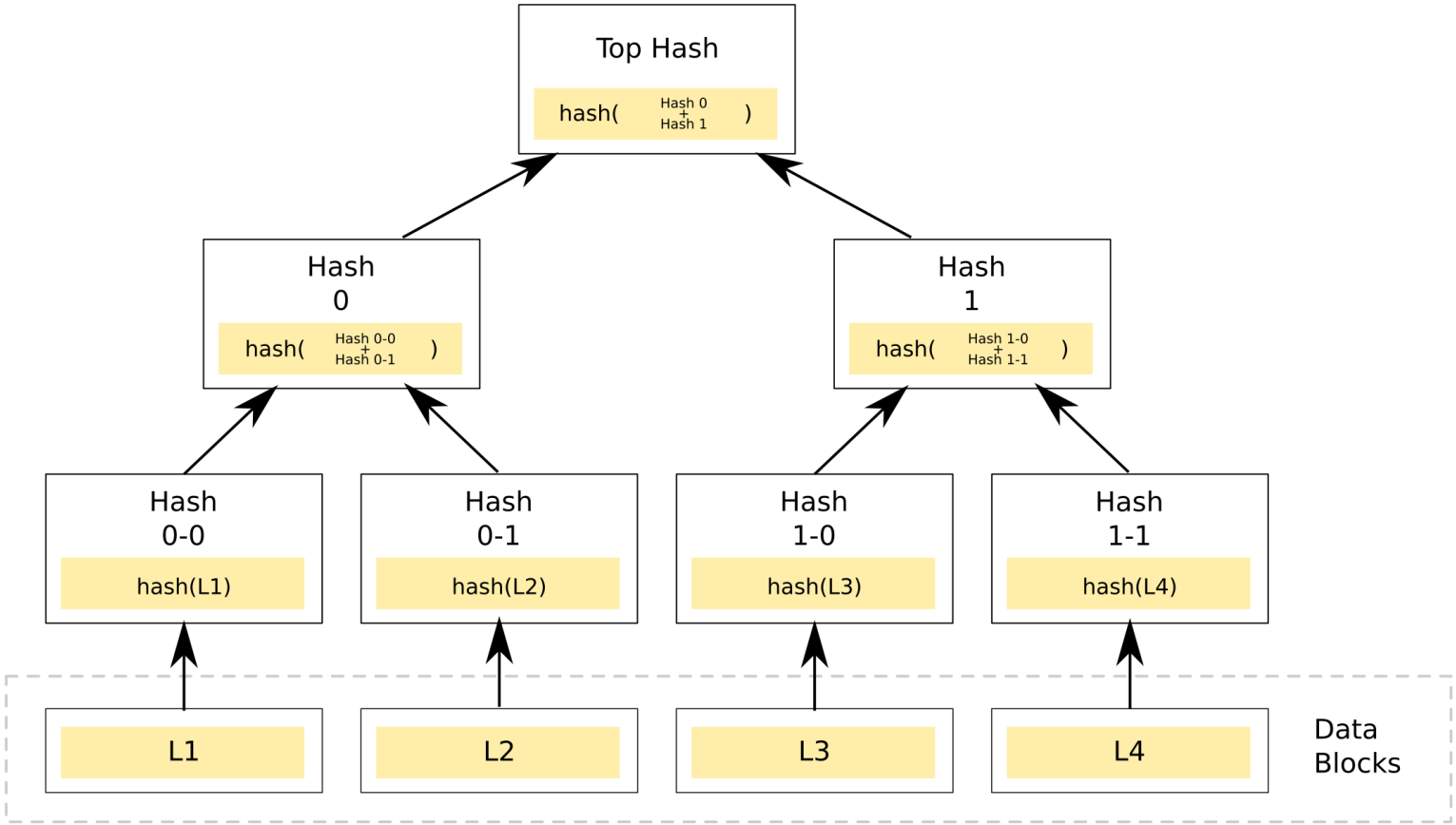
Top Hash
hash(Hash 0 + Hash 1)
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Hash 1
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Hash 0-0
hash(L1)
Hash 0-1
hash(L2)
Hash 1-0
hash(L3)
Hash 1-1
hash(L4)
L1
L2
L3
L4
Data Blocks